[This question paper contains 4 printed pages.]

Your Roll No.............

Sr. No. of Question Paper : 4548 **E**

Unique Paper Code : 32351201

Name of the Paper : Real Analysis (CBCS-LOCF)

Name of the Course : **B.Sc. (Hons) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any two parts from each question.

1. (a) If x and y are positive real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. (6.5)

   (b) Define Infimum and Supremum of a nonempty set of $\mathbb{R}$. Find infimum and supremum of the set

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$ (6.5)

P.T.O.

(c) State the completeness property of $\mathbb{R}$, hence show that every nonempty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$. (6.5)

2. (a) Prove that there does not exist a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$. (6)

(b) Define an open set and a closed set in $\mathbb{R}$. Show that if $a, b \in \mathbb{R}$, then the open interval (a, b) is an open set. (6)

(c) Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as: s \in S\}$. Prove that $\inf(aS) = a(\inf S)$ and $\sup(aS) = a(\sup S)$. (6)

3. (a) Define limit of a sequence. Using definition show that $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$. (6.5)

(b) Prove that every convergent sequence is bounded. Is the converse true? Justify. (6.5)

(c) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \in \mathbb{N}$. Show that $\langle x_n \rangle$ is bounded and monotone. Find the limit. (6.5)

4. (a) If $\langle a_n \rangle$ and $\langle b_n \rangle$ converges to a and b respectively, prove that $\langle a_n b_n \rangle$ converges to ab. (6)

(b) Show that $\lim_{n\to\infty} n^{1/n} = 1$. (6)

(c) State Cauchy Convergence Criterion for sequences. Hence show that the sequence $\langle a_n \rangle$, defined by $a_n = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{n}$, does not converge. (6)

5. (a) Prove that if an infinite series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$. Hence examine the convergence of $\sum_{n=1}^{\infty} \frac{n}{2n+3}$. (6.5)

(b) Examine the convergence or divergence of the following series.

(i) $\frac{2}{5} + \frac{4}{8} + \frac{6}{11} + \cdots$ (6.5)

(ii) $\sum_{n=1}^{\infty}\left(\frac{3n+5}{2n+1}\right)^{n/2}$

(c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$, $p>0$ is convergent for $p>1$ and divergent for $p \le 1$. (6.5)

6. (a) State and prove ratio test (limit form). (6)

(b) Examine the convergence or divergence of the following series. (6)

(i) $\sum_{n=1}^{\infty}\frac{n^3+1}{n^4+3n^2+2n}$

(ii) $3+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots$

(c) Prove that the series $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\cdots$ is conditionally convergent. (6)

[This question paper contains 2 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4686** **E**

Unique Paper Code : 32351402

Name of the Paper : Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **two** parts from each question.

1(a) Let f be integrable on [a, b], and suppose g is a function on [a, b] such that $g(x) = f(x)$ except for finitely many x in [a, b]. Show g is integrable and $\int_a^b g = \int_a^b f$ (6)

(b) Show that if f is integrable on [a, b] then f^2 also is integrable on [a, b]. (6)

(c) (i) Let f be a continuous function on [a,b] such that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)\,dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (3)

(ii) Give an example of function such that $|f|$ is integrable on [0,1] but f is not integrable on [0,1]. Justify it. (3)

2(a) State and prove Fundamental Theorem of Calculus I. (6.5)

(b) State Intermediate Value Theorem for Integrals. Evaluate $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$. (6.5)

(c) Let function $f: [0,1] \to R$ be defined as

$$f(x) = \begin{cases} x^2 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval [0,1]. Is f integrable on [0.1] ? (6.5)

3(a) Examine the convergence of the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$. (6)

(b) Show that the improper integral $\int_\pi^\infty \frac{Sinx}{x}dx$ is convergent but not absolutely convergent. (6)

P.T.O.

(c) Determine the convergence or divergence of the improper integral (6)

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_1^\infty \frac{xdx}{\sqrt{x^3+x}}$

4(a) Show that the sequence

$$f_n(x) = \frac{nx}{1+nx}, \quad x \in [0,1], \quad n \in N$$

converges non-uniformly to an integrable function f on [0,1] such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$ (6.5)

(b) Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, \infty)$. (6.5)

(c) Let $\langle f_n \rangle$ be a sequence of continuous function on $A \subset R$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \longrightarrow R$. Show that f is continuous on A. (6.5)

5(a) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$. Show that sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty), a > 0$. (6.5)

(b) State and prove Weierstrass M-test for the uniform Convergence of a series of functions. (6.5)

(c) Show that the series of functions $\sum \frac{1}{n^2+x^2}$, converges uniformly *on* R to a continuous function. (6.5)

6(a) (i) Find the exact interval of convergence of the power series (3)

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

(ii) Define *sinx* as a power series and find its radius of convergence (3)

(b) Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for $|x| < 1$ and hence evaluate $\frac{\sum_{n=1}^{\infty} n^2(-1)^n}{3^n}$. (6)

(c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad for \ |x| < R.$$ (6)

3 P

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4810 **E**

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra – I

Name of the Course : **B.Sc. [Hons.] Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question.

P.T.O.

1. (a) Find all the zero divisors and units in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. (6)

(b) Prove that characteristic of an integral domain is 0 or prime number p. (6)

(c) State and prove the Subring test (6)

2. (a) Let R be a commutative ring with unity and let A be an ideal of R then prove that R/A is a field if and only if A is a maximal ideal of R. (6)

(b) Let A and B are two ideals of a commutative ring R with unity and A+B=R then show that $A \cap B = AB$. (6)

(c) If an ideal I of a ring R contains a unit then show that I=R. Hence prove that the only ideals of a field F are {0} and F itself. (6)

3. (a) Find all ring homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_{15}$. (6.5)

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ and Φ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to a-b. Show that

(i) Φ is a ring homomorphism.

(ii) Determine Ker Φ.

(iii) Show that R/Ker Φ is isomorphic to $\mathbb{Z}$. (6.5)

(c) Using homomorphism, prove that an integer n with decimal representation $a_k a_{k-1} \dots \dots a_0$ is divisible by 9 iff $a_k + a_{k-1} + \dots + a_0$ is divisible by 9.

(6.5)

4. (a) Let V(F) be the vector space of all real valued function over $\mathbb{R}$.

Let $V_e = \{f \in V \mid f(x) = f(-x)\ \forall\ x \in \mathbb{R})$

and $V_o = \{f \in V \mid f(-x) = -f(x)\ \forall\ x \in \mathbb{R})$

Prove that V_e and V_0 are subspaces of V and $V = V_e \oplus V_o$. (6)

(b) Let V(F) be a vector space and let $S_1 \subseteq S_2 \subseteq V$.

Prove that

(i) If S_1 is linearly dependent then S_2 is linearly dependent

(ii) If S_2 is linearly independent then S_1 is linearly independent (6)

(c) Show that $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ forms a basis for $M_{2X2}(\mathbb{R})$. (6)

5. (a) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

Find Null space and Range space of T and verify Dimension Theorem. (6.5)

(b) Define $T: M_{2X2}(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$ be basis of $M_{2X2}(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Compute $[T]_\beta^\gamma$. (6.5)

(c) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For $w_1, w_2, \ldots, w_n$ in W. Prove that there exists exactly one linear transformation $T: V \rightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$. (6.5)

6. (a) Let T be the linear operator on $\mathbb{R}^2$ define by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Find $[T]_{\beta'}$. (6.5)

(b) Let V and W be finite dimensional vector spaces with ordered basis β and γ respectively. Let T: V → W be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. (6.5)

(c) Let V, W and Z be finite dimensional vector spaces with ordered basis α, β, γ respectively. Let T: V → W and U: W → Z be linear transformations.

Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$. (6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4512 **E**

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt **two** parts from each question.

1. (a) Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \text{ and } 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$. (6)

P.T.O.

(b) (i) Prove that the limit of the function

$$f(z)=\left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

(ii) Show that

$$\lim_{z\to\infty}\frac{4z^2}{(z-1)^2}=4.$$ (3+3=6)

(c) Show that the following functions are nowhere differentiable.

(i) $f(z) = z - \bar{z}$,

(ii) $f(z) = e^y \cos x + ie^y \sin x$. (3+3=6)

(d) (i) If a function f(z) is continuous and nonzero at a point z_0, then show that $f(z) \neq 0$ throughout some neighborhood of that point.

(ii) Show that the function $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire. (3+3=6)

2. (a) (i) Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$|\exp \exp (2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}$.

(ii) Find the value of z such that

$$e^z = 1+\sqrt{3}i$$ (3.5+3=6.5)

(b) Show that

(i) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

(ii) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n = 0 \pm 1, \pm 2, \ldots.)$. (3.5+3=6.5)

(c) Show that

(i) log log (i^2) = 2logi where

$$\log z = \ln r + i\theta (r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}).$$

(ii) log log $(i^2) \neq$ 2logi where

$$\log z = \ln r + i\theta (r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}).$$

(3.5+3=6.5)

(d) Find all zeros of sinz and cosz. (3.5+3=6.5)

3. (a) State Fundamental theorem of Calculus.

Evaluate the following integrals to test if Fundamental theorem of Calculus holds true or not :

(i) $\int_0^{\pi/2} \exp(t+it)dt$

(ii) $\int_0^1 (3t-i)^2 dt$ (2+2+2=6)

(b) Let y(x) be a real valued function defined piecewise on the interval $0 \le x \le 1$ as

$$y(x) = x^3\sin(\pi/x),\ 0 < x \le 1 \text{ and } y(0) = 0$$

Does this equation $z = x + iy,\ 0 \le x \le 1$ represent

(i) an arc

(ii) A smooth arc. Justify.

Find the points of intersection of this arc with real axis. (2+2+2=6)

(c) For an arbitrary smooth curve C: $z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integral depends only on the end points of C.

State if it is independent of the arc under consideration or not?

Also, find its value around any closed contour.

(3+1+2=6)

(d) Without evaluation of the integral, prove that

$\left|\int_c \frac{1}{z^2+1}dz\right| \le \frac{1}{2\sqrt{5}}$ where C is the straight line segment from 2 to 2 +i. Also, state the theorem used.

(4+2=6)

4. (a) Use the method of antiderivative to show that

$\int_c (z-z_0)^{n-1} dz = 0$, n = ±1, ±2,-- where C is any closed contour which does not pass through the point z_0. State the corresponding result used.

(4+2.5=6.5)

(b) Use Cauchy Gourset theorem to evaluate :

(i) $\int_c f(z)dz$, when $f(z)=\frac{1}{z^2+2z+2}$ and C is the unit circle $|z| = 1$ in either direction.

(ii) $\int_c f(z)dz$, when $f(z) = \dfrac{5z+7}{z^2+2z-3}$ and C is the circle $|z-2| = 2$. (3+3.5=6.5)

(c) State and prove Cauchy Integral Formula. (2+4.5=6.5)

(d) Evaluate the following integrals :

(i) $\int_c \dfrac{\cos z}{z(z^2+8)} dz$, where C is the positive oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(ii) $\int_c \dfrac{2s^2 - s - 2}{s - 2} ds$, $|z| \neq 3$ at $z = 2$, where C is the circle $|z| = 3$. (3.5+3=6.5)

5. (a) If a series of complex numbers converges then prove that the nth term converges to zero as n tends to infinity. Is the converse true? Justify. (6.5)

(b) Find the Maclaurin series for the function $f(z) = \sinh z$. (6.5)

(c) If a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in powers of $z - z_0$. (6.5)

(d) Find the integral of f(z) around the positively oriented circle $|z| = 3$ when $f(z)=\dfrac{(3z+2)^2}{z(z-1)(2z+5)}$. (6.5)

6. (a) For the given function $f(z)=\left(\dfrac{z}{2z+1}\right)^3$, show any singular point is a pole. Determine the order of each pole and find the corresponding residue. (6)

(b) Find the Laurent Series that represents the function $f(z)=z^2 \sin\dfrac{1}{z^2}$ in the domain $0 < |z| < \infty$. (6)

(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,...)$ and $S = X + iY$. Then show that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

(6)

(d) If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that

$$\int_c f(z)dz = 2\pi i\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)

[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 4792 **E**

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : **B.Sc. (H) Mathematics (CBCS – LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated

1. (a) (i) Prove that If F is a field, then F[x] is a Principal Ideal Domain.

   (ii) Is $\mathbb{Z}[x]$, a Principal Ideal Domain? Justify your answer.

   (b) Prove that $<x^2 + 1>$ is not a maximal ideal in $\mathbb{Z}[x]$.

P.T.O.

(c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)

2. (a) (i) State and prove Gauss's Lemma.

(ii) Is every irreducible polynomial over $\mathbb{Z}$ primitive? Justify.

(b) Construct a field of order 25.

(c) In $\mathbb{Z}\left[\sqrt{(-5)}\right]$, prove that $1+3\sqrt{(-5)}$ is irreducible but not prime. (4+2.5,6.5,6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$f_1(x, y, z) = x - 2y,$

$f_2(x, y, z) = x + y + z,$

$f_3(x, y, z) = y - 3z.$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and then find a basis for V for which it is the dual basis.

(b) Test the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(0) + f(1)(x + x^2)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Find an expression for A^n where n is an arbitrary natural number. (6,6,6)

4. (a) For a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (-b + c, a + c, 3c)$, determine the T-cyclic subspace W of $\mathbb{R}^3$ generated by $e_1 = (1, 0, 0)$. Also find the characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theorem and verify it for the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

(c) Show that the vector space $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$ where $W_1 = \{(a, b, 0, 0): a, b \in \mathbb{R})$, $W_2 = \{(0,0, c, 0): c \in \mathbb{R}\}$ and $W_3 = \{(0,0,0, d): d \in \mathbb{R}\}$.

(6.5,6.5,6.5)

5. (a) Consider the vector space $\mathbb{C}$ over $\mathbb{R}$ with an inner product $<.,.>$. Let $\bar{z}$ denote the conjugate of z. Show that $<.,.>'$ defined by $<z, w>' = <\bar{z}, \bar{w}>$ for all $z, w \in \mathbb{C}$ is also an inner product on $\mathbb{C}$. Is $<.,.>''$ defined by $<z, w>'' = <z + \bar{z}, w + \bar{w}>$ for all $z, w \in \mathbb{C}$ an inner product on $\mathbb{C}$? Justify your answer.

(b) Let $V = P(\mathbb{R})$ with the inner product $<p(x), q(x)> = \int_{-1}^{1} p(t)q(t)dt \ \forall p(x), q(x) \in V$. Compute the orthogonal projection of the vector $p(x) = x^{2k-1}$ on $P_2(\mathbb{R})$, where $k \in \mathbb{N}$.

(c) (i) For the inner product space $V = P_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator T on V defined by $T(f) = f' + 3f$, compute $T^*(4 - 2t)$.

(ii) For the standard inner product space $V = \mathbb{R}^3$ and a linear transformation $g: V \to \mathbb{R}$ given by $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$, find a vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

(6,6,2+4)

6. (a) Prove that a normal operator T on a finite-dimensional complex inner product space V yields an orthonormal basis for V consisting of eigenvectors of T. Justify the validity of the conclusion of this result if V is a finite-dimensional real inner product space.

(b) Let $V = M_{2\times2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) For the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

(6.5,6.5,6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

F

Sr. No. of Question Paper : 1204

Unique Paper Code : 2352011201

Name of the Paper : Linear Algebra

Name of the Course : **B.Sc. (H) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting **two** parts from each question.

3. **All** questions carry equal marks.

4. Use of Calculator not allowed.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that $||x + y|| \leq ||x|| + ||y||$. Also, verily the same for the vectors x = [–1, 4, 2, 0, –3] and y = [2, 1, –4, –1, 0] in $\mathbb{R}^5$.

(b) Using the Gauss – Jordan method, find the complete solution set for the following homogeneous system of linear equations:

$$4x_1 - 8x_2 - 2x_3 = 0$$

$$3x_1 - 5x_2 - 2x_3 = 0$$

$$2x_1 - 8x_2 + x_3 = 0$$

(c) Define the rank of a matrix. Using rank, find whether the non-homogeneous linear system AX = B, where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

has a solution or not. If yes, find the solution.

2. (a) Consider the matrix :

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 5 \\ -4 & -3 & 3 \end{pmatrix}$$

Determine whether the vector [4, 0, –3] is in the row space of A. If so, then express [4, 0, –3] as a linear combination of the rows of A.

(b) Consider the matrix :

$$A = \begin{pmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{pmatrix}$$

(i) Find the eigenvalue and the fundamental eigenvectors of A.

(ii) Is A diagonalizable? Justify your answer.

(c) Find the reduced row echelon form matrix B of the following matrix :

$$A = \begin{pmatrix} 1 & 2 & -2 & -11 \\ 2 & 4 & -1 & -10 \\ 3 & 6 & -4 & -25 \end{pmatrix}$$

and then give a sequence of row operations that convert B back to A.

3. (a) Let F_1 and F_2 be fields. Let $\mathcal{F}(F_1, F_2)$ denote the vector space of all functions from F_1 to F_2. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd

function if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

(b) Let W_1 and W_2 be subspaces of a vector space V.

(i) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(ii) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(c) (i) Let S_1 and S_2 are arbitrary subsets of a vector space V. Show that if $S_1 \subseteq S_2$ then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(ii) Let F be any field. Show that the vectors (1,1.0), (1,0,1) and (0,1,1) generate F^3.

4. (a) Define a linearly independent subset of a vector space V. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k, $(1 \leq k < n)$.

(b) Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Prove that β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$, for unique scalars $a_1, a_2, \ldots, a_n$.

(c) Let F be any field. Consider the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$$

Find bases and dimension for the subspaces W_1, W_2 and $W_1 \cap W_2$.

5. (a) Let V and W be vector spaces over a field F, and let $T : V \to W$ be a linear transformation. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then prove that

$R(T) = \text{span}\ (T(\beta)) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\}$

If T is one-to-one and onto then prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear,

$T(1,1) = (1, -2)$

$T(-1, 1) = (2, 3).$

What is $T(-1,5)$ and $T(x_1, x_2)$?

Find $[T]_\beta^\gamma$ if $\beta = \{(1,1), (-1,1)\}$ and $\gamma = \{(1,-2), (2,3)\}$.

(c) For the following linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

find bases for null space N(T) and range space R(T). Also, verify the dimension theorem.

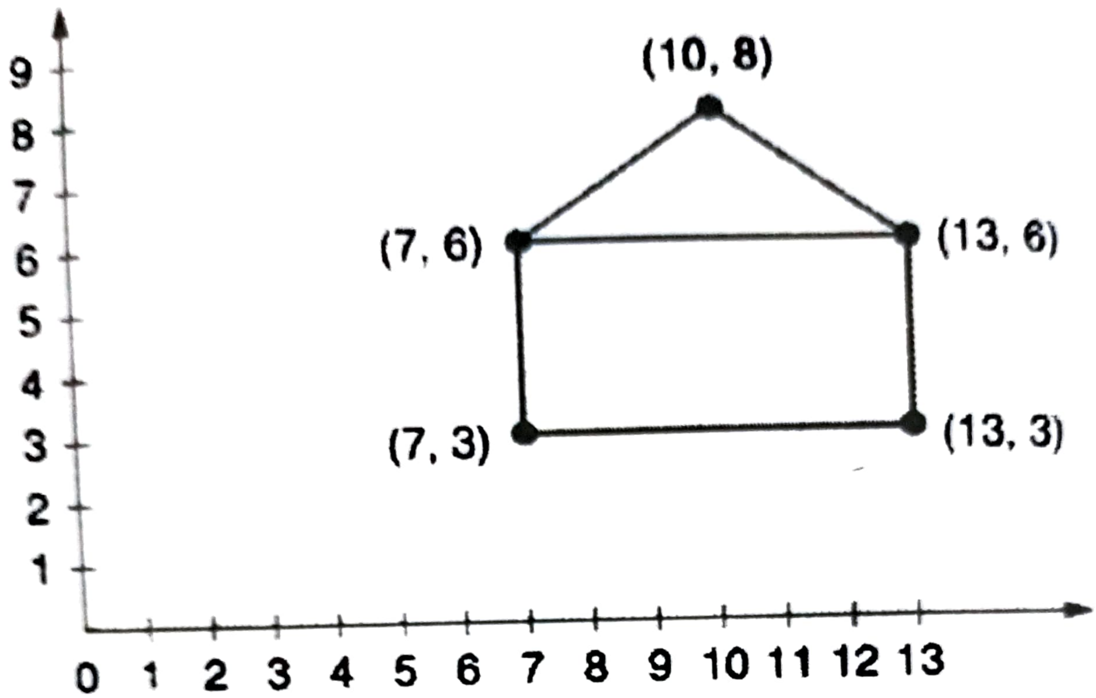
6. (a) Let V and W be finite dimensional vector spaces over the same field F. Then, prove that V is isomorphic to W if and only if dim V = dim W. Are $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$ isomorphic? Justify your answer.

(b) Let V and W be vector spaces and let T: V → W be linear and invertible. Prove that T^{-1}: W → V is linear. For the linear transformation T: $M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by :

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

determine whether T is invertible or not. Justify your answer.

(c) For the adjoining graphic, use homogenous co-ordinates to find the new vertices after performing scaling about (7,3) with scale factors of ½ in the x - direction and 3 in the y - direction.



Also, sketch the final figure that would result from this movement.

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1223 **F**

Unique Paper Code : 2352011202

Name of the Paper : CALCULUS

Name of the Course : **B.Sc. (H) Mathematics UGCF-2022**

Semester : II – DSC 5

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.

1. (a) State and prove the sequential criterion for the limit of a real valued function. (5)

(b) Use $\in - \delta$ definition of limit to establish the following limit : (5)

$$\lim_{x \to 2} \frac{1}{1-x} = -1.$$

P.T.O.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as (5)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f has a limit only at $x = 0$.

(d) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f > 0$, then show that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

2. (a) If f is continuous at x_0 and g is continuous at $f(x_0)$ then prove that the composite function $g \circ f$ is continuous at x_0. (5)

(b) Let $f(x) = \frac{1}{x}\sin\frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is discontinuous at 0. (5)

(c) State Intermediate Value Theorem. Prove that $xe^x = 1$ for some x in $(0,1)$. (5)

(d) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$. (5)

3. (a) Prove that if a real valued function f is continuous on $[a, b]$ then it is uniformly continuous on $[a, b]$. (5)

(b) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞) for $a > 0$ but it is not uniformly continuous on $(0,1)$. (5)

(c) Let $f(x) = |x| + |x - 1|$, $x \in \mathbb{R}$. Draw the graph and give the set of points where it is not differentiable. Justify also. (5)

(d) Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$. (5)

4. (a) State and prove Mean Value Theorem. (5)

(b) State Intermediate Value Theorem for derivatives. Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.

(ii) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0,2)$.

(5)

(c) Prove that $(\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (5)

(d) Let f be defined on $\mathbb{R}$ and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. (5)

5. (a) Let f be differentiable function on an open interval (a, b). Then show that f is increasing on (a, b) if $f'(x) \geq 0$. (5)

(b) If $y = e^{\tan^{-1}x}$, prove that (5)

$$(1 + x^2)y_{n+2} + (2(n+1)x - 1)y_{n+1} + n(n+1)y_n = 0.$$

(c) If $y = \cos(m \sin^{-1} x)$, find $y_n(0)$. (5)

(d) Stating Taylor's theorem find Taylor series expansion of e^x. (5)

6. (a) Find $\lim_{x \to +\infty} \left[x - \ln\left(x^2 + 1\right)\right]$. (5)

(b) Determine the position and nature of the double points on the curve (5)
$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

(c) Sketch a graph of the rational function showing the horizontal, vertical and oblique asymptote (if any) of $y = \dfrac{x^2 - 2}{x}$. (5)

(d) Sketch the curve in polar coordinates of $r = \sin 2\theta$. (5)

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[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1242 **F**

Unique Paper Code : 2352011203

Name of the Paper : Ordinary Differential Equations

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts of each question.

3. Each part carries **7.5** marks.

4. Use of non-programmable Scientific Calculator is allowed.

1. (a) Solve the initial value problem

$$(e^{2x}y^2 - 2x)\,dx + e^{2x}y\,dy = 0,\ y(0) = 2$$

(b) Solve

$$(2x + \tan y)\, dx + (x - x^2 \tan y)\, dy = 0$$

(c) Solve

(i) $(3x^2 + 4xy - 6)\, dy + (6xy + 2y^2 - 5)\, dx = 0$

(ii) $\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right)^3 = 2y$ by reducing the order.

2. (a) A certain rumor began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population has heard the rumor?

(b) The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident in a certain region has left the level of cobalt to be 100 times the acceptable level for habitation. How long will it be until the region is again habitable?

(c) A cake is removed from an oven at 210°F and left to cool at room temperature of 70°F. After 30 minutes, the temperature of the cake is 140°F. What will be its temperature after 40 minutes? When will the temperature be 100°F?

3. (a) Show that the solutions x, x^2, x log x of the third order differential equation

$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$$

are linearly independent on (0, ∞). Also find the particular solution satisfying the given initial condition.

$$y(1) = 3,\ y'(1) = 2,\ y''(1) = 1$$

(b) Solve the differential equation using the method of Variation of Parameters

$$\frac{d^2y}{dx^2} + 9y = \tan 3x$$

(c) Find the general solution of the differential equation using the method of undetermined Coefficients.

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 4e^{-x} + 3x^2$$

4. (a) Use the operator method to find the general solution of the following linear system

$$2\frac{dx}{dt} + \frac{dy}{dt} + x + 5y = 4t$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + 2y = 2$$

(b) Solve the initial value problem. Assume x > 0.

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 8y = 2x^3, \quad y(2) = 0, \; y'(2) = 8$$

(c) A body with mass $m = \frac{1}{2}$ kg is attached to the end of a spring that is stretched 2m by a force of 16N. It is set in motion with initial position $x_0 = 1$m and initial velocity $v_0 = -5$m/s. Find the position function of the body as well as the amplitude, frequency and period of oscillation.

5. (a) Define the term Carrying Capacity. Derive the logistic equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$$

where K is the carrying capacity of the population. Also find the solution.

(b) The per-capita death rate for the fish is 0.5 fish per day per fish, and the per-capita birth rate is 1.0 fish per day per fish. Write a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the number of fish. If the fish population at a given time is 240, 000, give an estimate of the number of fish born in one week.

(c) In an epidemic model where infected get recovered, the differential equation is of the form

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I$$

Use parameter values $\beta = 0.002$ and $\gamma = 0.4$, and assume that initially there is only one infective but there are 500 susceptibles. How many susceptibles never get infected, and what is the maximum number of infectives at any time? What happens as time progresses, if the initial number of susceptibles is doubled, $S(0) = 1000$? How many people were infected in total.

6. (a) A public bar opens at 6 p.m. and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators that exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20m by 15m, and a height of 4m. It is estimated that smoke enters the room at a constant

rate of 0.006 m^3/min, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a good time to leave the bar, that is, sometime before the concentration of carbon monoxide reaches the lethal limit. Starting from a word equation or a compartmental diagram, formulate the differential equation for the changing concentration of carbon monoxide in the bar over time. By solving the equation above, establish at what time the lethal limit will be reached.

(b) Find the equilibrium solution of the differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$$

And discuss the stability of equilibrium solution.

(c) Consider a disease where the infected get recovered. A model describing this is given by the differential equations

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I$$

Use chain rule to find arelation between S and I. Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.

[This question paper contains 12 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4748 **E**

Unique Paper Code : 32357609

Name of the Paper : DSE-3 : Bio-Mathematics

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Use of Scientific Calculator is allowed.

1. (a) A doctor has to prescribe medicine to the patient. The medicine raises the blood plasma concentration of an average adult by 20 mg l^{-1} and takes 6 h to decay in the blood plasma. The maximum

permissible limit of concentration of drug in the body is 40 mg l^{-1}. What time gap he will ensure for maintaining a safe decomposition of drug. Find out the minimum concentration if doses are given at this time interval? The concentration of another drug, decreases by 40% in 20 h. Find how long will it take for this drug to fall to 5% of its initial value. (6)

(b) Observation on animal tumours indicate that theirsizes obey the Gompertz growth law $\frac{ds}{dt} = ks\ln\left(\frac{S}{s}\right)$ rather than the logistic law. Here k and S are positive constants. By putting y = ln(s), prove that $s(t) = Se^{-Ae^{-kt}}$, where $A = \ln\left(\frac{S}{s_0}\right)$, s_0

being the size at $t = 0$. Discuss the model describing drug concentration and residual concentration at any time nto, in which drug decays according to equation $\frac{dc}{dt} = \frac{-c(t)}{\tau}$. When dose is administered regularly at time $t = 0, t_0, 2t_0, 3t_0, \ldots$ with assumption that each dose raises the drug concentration by fixed amount Co. Find the maximum possible concentration and residue as n increases. (6)

(c) Consider the following chemical reaction, with the rate constant as q :

$$P + X \rightarrow 7Y.$$

If the reactant P is held at a constant concentration p, derive a system of equations for the concentrations of X and Y. Suppose the initial concentrations of X and Y are X_0 and Y_0 respectively. Solve the system of equations to obtain X(t) and Y(t). (6)

(d) Show that Zeeman's heartbeat equations have a unique resting state

$$x = x_a, \quad b = -(x_a^3 + a\,x_a).$$

Then derive the single differential equation satisfied by muscle fibre of length x. (6)

2. (a) Discuss the nature of fixed point and give equation of trajectories for the given system

$$\dot{x} = 6x + 12y$$
$$\dot{y} = 3x + y$$
(6½)

(b) (i) Discuss the two equilibrium states during the heart beat cycle and the role of pacemaker in the heartbeat cycle.

(ii) Discuss the threshold level and firing of axon in nerve impulse transmission. (6½)

(c) Examine the possibility of periodic solutions of

$$c\ddot{x} + (2 + 3ax + 4bx^2)x = 0$$

Where a, b and c are constants, c being positive. (6½)

(d) Describe the epidemic model and show that population return to equilibrium after the small departure from the equilibrium. (6½)

3. (a) Consider the system

$$\frac{du}{dt} = u\,(1 - u)\,(u - a) - w$$

$$\frac{dw}{dt} = bu - \gamma w.$$

Where $0 < a < 1$, $b > 0$, $\gamma \geq 0$.

Linearise the above system about (0,0). Further assuming that $u = \alpha e^{\lambda t}$, $w = \beta e^{\gamma t}$ be solutions of the linearised system about (0,0), show that the rest state (0,0) is locally stable. (6)

(b) Sketch the trajectories of the following system

$$\dot{x} = y$$

$$\dot{y} = \frac{1}{2}(1 - x^2) \qquad (6)$$

(c) Define

(i) Bifurcation

(ii) Bifurcation Point

Make the sketches for Pitchfork bifurcation, Saddle-node bifurcation and Hopf bifurcation.

(6)

(d) For the iteration scheme $x_{n+1} = \mu x_n(1 - x_n)$, $n \geq 1$,

$$x_0 = \lim_{n \to \infty} x_n$$

Show that there are bifurcations at $\mu = 1$ and $\mu = 3$. (6)

4. (a) Find the constraints on a, b and λ assuming it has a unique rest state, taking the solutions to the travelling wave equations in the form $u = \phi(x + ct)$, $w = \psi(x + ct)$ of the following system of equations $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) - w$,

$\frac{\partial w}{\partial t} = bu - \gamma w.$ (6½)

(b) What is Flopf bifurcation? Show that Hopf bifurcation holds for the following system

$$\dot{x} = -y + x(\mu - x^2 - y^2)$$

$$\dot{y} = x + y(\mu - x^2 - y^2)$$ (6½)

(c) Provide a full phase plane analysis for the mathematical model of heart beat equations given by

$$\varepsilon\frac{dx}{dt} = -\left(x^3 - Tx + b\right), \qquad T > 0$$

$$\frac{db}{dt} = x - x_0 .$$

Where x is the muscle fibre length, b is the chemical control, $\varepsilon > 0$ and (x_0, b_0) is a rest state. (6½)

(d) Show that the following system has limit cycle.

$$\frac{du}{dt} = u(1-u)(u-a) - w + I(t),$$

$$\frac{dw}{dt} = bu - \gamma w,\ 0 < a < 1,\ b > 0,\ \gamma \geq 0. \qquad (6\tfrac{1}{2})$$

5. (a) Write down the steps of Neighbor Joining Algorithm. From the given distance table of four taxa S_1, S_2, S_3 and S_4, compute R_1, R_2, R_3, R_4 and then form a table of values for $M(S_i, S_j)$ for $1 \leq i \neq j \leq 4$.

	S_1	S_2	S_3	S_4
S_1		1.2	0.9	1.7
S_2			1.1	1.9
S_3				1.6

(6½)

(b) If D and d denote the alleles for tall and dwarf plant and if W and w denote the alleles for round and wrinkled seed, then create a Punnett square for a DdWw × ddWw cross pea plant and compute the probability of a tall plant with wrinkled seeds. (6½)

(c) Derive the formula for the Jukes-Cantor distance (d_{JC}) given that all the diagonal entries of Jukes Cantor matrix M^t are $\frac{1}{4}+\frac{3}{4}\left(1-\frac{4}{3}\alpha\right)^t$, where a is the mutation rate. Compute the Jukes-Cantor distance $d_{JC}(S_0,S_1)$ to 4 decimal digits, from the following 40 base table : (6½)

$S_1 \backslash S_0$	A	G	C	T
A	7	0	1	1
G	1	9	2	0
C	0	2	7	2
T	1	0	1	6

(d) Describe when two trees are considered to be topologically similar. Draw all topologically distinct un-rooted bifurcation trees that could describe the relationship between 3 taxa and 4 taxa. (6½)

6. (a) Define phylogenetic tree, bifurcating tree and unrooted tree with examples of each. (6)

(b) Explain Kimura 2-parameter and 3-parameter models along with their corresponding distance formulas. Write the expression of the log-det distance between S_0 and S_1. (6)

(c) In mice, an allele A for agouti- or gray-brown grizzled fur is dominant over the allele a, which determines a non-agouti color. If an Aa×Aa cross produces 4 offsprings, then compute the probabilities that :

(i) No offspring have agouti fur.

(ii) Exactly 3 of 4 offspring have agouti fur.

(6)

(d) From the given distance table of four sequences S_1, S_2, S_3 and S_4 of DNA, construct a rooted tree showing the relationship between S_1, S_2, S_3 and S_4 by UPGMA

	S_1	S_2	S_3	S_4
S_1		0.45	0.27	0.53
S_2			0.40	0.50
S_3				0.62

(6)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4749 **E**

Unique Paper Code : 32357614

Name of the Paper : DSE-3 MATHEMATICAL FINANCE

Name of the Course : **B.Sc. (H) Mathematics CBCS (LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory and carry equal marks.
4. Use of Scientific calculator, Basic calculator and Normal distribution tables all are allowed.

1. (a) Explain Duration of a zero-coupon bond. A 5-year bond with a yield of 12% (continuously compounded) pays a 10% coupon at the end of each year.

(i) What is the bond's price?

(ii) Use duration to calculate the effect on the bond's price of a 0.1% decrease in its yield? (You can use the exponential values: e^x = 0.8869, 0.7866, 0.6977, 0.6188, and 0.5488 for x = –0.12, –0.24, –0.36, –0.48, and –0.60, respectively)

(b) Portfolio A consists of 1-year zero coupon with a face value of ₹2000 and a 10-year zero coupon bond with face value of ₹6000. Portfolio B consists of a 5.95-year zero coupon bond with face value of ₹5000. The current yield on all bonds is 10% per annum.

(i) Show that both portfolios have the same duration.

(ii) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

(You can use the exponential values: e^x = 0.905, 0.368, 0.552, 0.861, 0.223 and 0.409 for x = –0.1, –1.0, –0.595, –0.15, –1.5 and –0.893 respectively)

(c) Explain difference between Continuous Compounding and Monthly Compounding. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

(d) (i) "When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain.

(ii) Why does loan in the repo market involve very little credit risk?

2. (a) Explain Hedging. How is the risk managed when Hedging is done using?

(i) Forward Contracts; (ii) Options

(b) (i) Suppose that a March call option to buy a share for ₹50 costs ₹2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised?

(ii) It is May, and a trader writes a September put option with a strike price of ₹20. The stock price is ₹18, and the option price is ₹2. Describe the trader's cash flows if the option is held until September and the stock price is ₹25 at that time.

(c) Write a short note on European put options. Explain the payoffs in different types of put option positions with the help of diagrams.

(d) (i) A trader writes an October put option with a strike price of ₹35. The price of the option is ₹6. Under what circumstances does the trader make a gain.

(ii) A company knows that it is due to receive a certain amount of a foreign currency in 6 months. What type of option contract is appropriate for hedging?

3. (a) Draw the diagrams illustrating the effect of changes in volatility and risk-free interest rate on both European call and put option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

(b) Derive the put-call parity for European options on a non-dividend-paying stock. Use put-call parity to derive the relationship between the vega of a European call and the vega of a European put on a non-dividend-paying stock.

(c) A European call option and put option on a stock both have a strike price of ₹20 and an expiration date in 3 months. Both sell for ₹3. The risk-free interest rate is 10% per annum, the current stock price is ₹19, and a ₹1 dividend is expected in 1 month. Identify the arbitrage opportunity open to a trader? ($e^{-0.0083} = 0.9917$)

(d) Find lower bound and upper bound for the price of a 1-month European put option on a non-dividend-paying stock when the stock price is ₹30, the strike price is ₹34, and the risk-free interest rate is 6% per annum? Justify your answer with no arbitrage arguments, ($e^{-0.005} = 0.9950$)

4. (a) Consider the standard one-period model where the stock price goes from S_0 to S_0u or S_0d with $d < 1 < u$, and consider an option which pays f_u or f_d in each case, and assume that the interest rate is r and time to maturity is T. Derive the formula for the no-arbitrage price of the option.

(b) A stock price is currently ₹40. It is known that at the end of one month it will be either ₹42 or ?38. The risk-free interest rate is 6% per annum with continuous compounding. Consider a portfolio consisting of one short call and Δ shares of the stock. What is the value of Δ which makes the portfolio riskless? Using no-arbitrage arguments, find the price of a one-month European call option with a strike price of ₹39? (You can use exponential value: $e^{0005} = 1.005$)

(c) Construct a two-period binomial tree for stock and European call option with

$S_0 = ₹100$, $u = 1.3$, $d = 0.8$, $r = 0.05$, $T = 1$ year, $K = ₹95$

and each period being of length $\Delta t = 0.5$ year. Find the price of the European call. If the call was American, will it be optimal to exercise the option early? Justify your answer. ($e^{-0.025} = 0.9753$)

(d) What do you mean by the volatility of a stock? How can we estimate volatility from historical prices of the stock?

5. (a) Let S_0 denote the current stock price, σ the volatility of the stock, r be the risk-free interest rate and T denote a future time. In the Black-Scholes model, the stock price S_T at time T in the risk-neutral world satisfies

$$\ln S_T \sim \phi\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a norma distribution with mean m and variance v.

Using risk-neutral valuation, derive the Black-Scholes formula for the price of a European call option on the underlying stock S, strike price K and maturity T.

(b) A stock price follows log normal distribution with an expected return of 16% and a volatility of 35%. The current price is ₹38. What is the probability that a European call option on the stock with an exercise price of ₹40 and a maturity date in six months will be exercised? (You can use values: $\ln(38) = 3.638$, $\ln(40) = 3.689$)

(c) What is the price of a European call option on a non-dividend-paying stock when the stock price is ₹69, the strike price is ₹70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

(You can use exponential values: $e^{-0.0144} = 0.9857$, $e^{-0.025} = 0.9753$)

(d) A stock price is currently ₹40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

6. (a) Discuss theta of a portfolio of options and calculate the theta of a European call option on a non-dividend-paying stock where the stock price is ₹49, the strike price is ₹50, the risk-free interest rate is 5% per annum and the time to maturity is 20 weeks, and the stock price volatility is 30% per annum. ($\ln(49/50) = -0.0202$)

(b) (i) Explain stop-loss hedging scheme.

(ii) What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

(c) Find the payoff from a butterfly spread created using call options. Also draw the profit diagram corresponding to this trading strategy.

(d) Companies X and Y have been offered the following rates per annum on a ₹5 million 10-year investment :

	Fixed rate	Floating rate
Company X	8.0%	LIBOR
Company Y	8.8%	LIBOR

Company X requires a fixed-rate investment; Company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and that will appear equally attractive to X and Y.

\\

[This question paper contains 4 printed pages.]

Your Roll No..............

E

Sr. No. of Question Paper : 4873

Unique Paper Code : 32357610

Name of the Paper : DSE-4 (Number Theory)

Name of the Course : **CBCS (LOCF) – B.Sc. (H) (Mathematics)**

Semester : VI

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts of each question.
4. Question Nos. **1** to **3**, each part carries **6.5** marks and Question Nos. **4** to **6**, each part carries **6** marks.

1. (a) Determine all solutions in the positive integers of the Diophantine equation

$$18x + 5y = 48.$$

(b) Using Euclidean algorithm and theory of linear Diophantine equation, divide 100 into two summands such that one is divisible by 7 and other by 11.

(c) Write a short note on Prime number theorem.

(d) If $ca \equiv cb \pmod{n}$ then prove that $a \equiv b \pmod{n/d}$, where $d = \gcd(c, n)$.

2. (a) Verify that $0, 1, 2, 2^2, 2^3 \ldots\ldots\ldots 2^9$ form a complete set of residues modulo 11, but that $0, 1^2, 2^2, 3^2 \ldots\ldots 10^2$ do not.

(b) Find the solutions of the system of congruences :

$$3x + 4y \equiv 5 \pmod{13}$$

$$2x + 5y \equiv 7 \pmod{13}.$$

(c) Use Fermat's theorem to verify that 17 divides $11^{104} + 1$.

(d) Find the remainder when $2(26!)$ is divided by 29.

3. (a) Let F and f be two number – theoretic functions related by the formula

$$F(n) = \sum_{d/n} f(d)$$

Prove $f(n) = \Sigma_{d|n}\mu(d)F(n/d) = \Sigma_{d|n}\mu(n/d)F(d)$.

(b) Verify that 1000! terminates in 249 zeros.

(c) Use Euler's theorem for any integer a, to prove that $a^{13} \equiv a \pmod{2730}$

(d) Prove that $\emptyset\,(2^n - 1)$ is a multiple of n for any $n > 1$.

4. (a) For any positive integer n, prove

$\emptyset(n) = n\Sigma_{d|n}\mu(d)/d$.

(b) Define primitive roots of an integer by an example and show that if $F_n = 2^{2^n} + 1$, $n > 1$, is a prime then 2 is not a primitive root of F_n.

(c) If p is a prime number and $d|p-1$, then show that there are exactly $\varphi(d)$ incongruent integers having order d modulo p.

(d) Determine all the primitive roots of the primes $p = 11, 19$, and 23, expressing each as a power of one of the roots.

5. (a) If gcd $(m, n) = 1$, where $m > 2$ and $n > 2$, then prove that the integer 'mn' has no primitive roots.

(b) Solve the quadratic congruence

$$3x^2 + 9x + 7 \equiv 0 \pmod{13}.$$

(c) Show that 3 is quadratic residue of 23, but a nonresidue of 31.

(d) Prove that there are infinitely many primes of the form 4k+1.

6. (a) Find the value of Legendre symbol (1234/4567).

(b) Solve the quadratic congruence

$$x^2 \equiv 23 \pmod{7^3}.$$

(c) Using the linear cipher $C \equiv 5P + 11 \pmod{26}$, encrypt the message NUMBER THEORY IS EASY.

(d) When the RSA algorithm is based on the key $(n, k) = (3233, 37)$, what is the recovery exponent for the cryptosystem?

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[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 5006 **E**

Unique Paper Code : 62354443

Name of the Paper : Analysis (LOCF)

Name of the Course : **B.A. (Prog.)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any two parts from each question.

4. **All** questions carry equal marks.

1. (a) Define limit point of a set $S \subseteq \mathbb{R}$. Find the limit points of the following sets :

(i) $\mathbb{N}$

(ii) $\mathbb{R}$

(b) Define closed set. Prove that the union of finite number of closed sets is closed set.

(c) If A and B are non-empty bounded above subsets of $\mathbb{R}$ ans $C = \{x + y | x \in A, y \in B]$ then show that : $Sup(C) = Sup(A) + Sup(B)$.

(d) Define neighborhood of a point and an open set. Give an example of each of the following :

(i) A non-empty set which is a neighborhood of each of its points with the exception of one point.

(ii) A non-empty set which is neither an open set nor a closed set.

(iii) A non-empty closed set which is not an interval.

(iv) A non-empty open set which is not an interval.

2. (a) Test the continuity of function

$$f(x)=\begin{cases}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & x\neq 0\\ 0, & x=0\end{cases},$$

at $x = 0$.

(b) Show that the function f defined by $f(x) = x^2$ is uniformly continuous on $[-2,2]$.

(c) Prove that the union of an arbitrary family of open sets is an open set.

(d) Show that every continuous function on a closed interval is bounded.

3. (a) Show that a sequence cannot converge to more than one limit.

(b) Show that the sequence $<a_n>$ defined by :

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{n+n}, \ \forall \ n \text{ converges.}$$

(c) State Cauchy convergence criterion for sequences and hence show that the sequence $<x_n>$ defined by :

$$x_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\cdots + \frac{1}{2n-1},$$

does not converge.

(d) Show that every convergent sequence is bounded but the converse is not true.

4. (a) State Leibnitz test for convergence of an alternating series : $\sum_{1}^{\infty}(-1)^{n-1}u_n \ \forall\, n$ and test the convergence and absolute convergence of the series :

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots\cdots\cdots.$$

(b) Check the convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{3.6.9.\cdots 3n}{7.10.13.\cdots(3n+4)}x^n \ (x>0).$$

(c) Show that the sequence $<x_n>$ defined by :

$x_1 = 1,\ x_{n+1} = \dfrac{3+2x_n}{2+x_n},\ n \geq 2$ is convergent. Also find $\lim_{n\to\infty} x_n$.

(d) Test the convergence of the series whose n^{th} term is $\left(\sqrt{n+1}-\sqrt{n}\right)$.

5. (a) If $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences of real numbers such that

$\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ then prove that :

$$\lim_{n\to\infty} (a_n b_n) = ab.$$

(b) Define Riemann integrability of a function. Show that x^2 is integrable on any interval $[0, k]$.

(c) Show that $\lim_{n\to\infty} \frac{1}{n}\left[1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots\cdots+n^{\frac{1}{n}}\right]=1$.

(d) Define the sum of a convergent series. Find the sum of the following series :

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots\ldots$$

6. (a) Test the convergence and absolute convergence of the series :

(i) $\sum \frac{(-1)^{n-1}}{n^2}$.

(ii) $\sum \frac{(-1)^{n-1}}{n\sqrt{n}}$.

(b) Let $<a_n>$ be a sequence defined by

$$a_1 = 1,\ a_{n+1} = \frac{(2a_n + 3)}{4},\quad \forall\, n \geq 1,$$

Prove that $<a_n>$ is bounded above and monotonically increasing. Also find $\lim_{n\to\infty} a_n$.

(c) Prove that every continuous function is integrable.

(d) Discuss the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{\sin nx + \cos nx}{n^{3/2}} .$$

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[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 5025 **E**

Unique Paper Code : 62354443

Name of the Paper : Analysis (LOCF)

Name of the Course : **B.A. (Prog.)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question.

4. **All** questions carry equal marks.

1. (a) Let S be a non empty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as: s \in S\}$. Prove that $\inf aS = a \inf S$, $\sup aS = a \sup S$.

(b) Define order completeness property of real numbers.

(c) Define limit point of a set. Show that the set $\mathbb{N}$ of natural numbers has no limit point.

(d) State and prove Archimedean property of real numbers.

2. (a) Show that the function defined as

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational} \\ -x, & \text{if } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

(b) Show that the function f defined by $f(x) = x^2$ is uniformly continuous on $[-2,2]$.

(c) Define an open set. Prove that every open interval is an open set. Which of the following sets are open.

(i) $]2, \infty[$

(ii) $[3,4[$

(d) Let A and B be bounded nonempty subsets of R, and let $A + B = \{a + b: a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$.

3. (a) Prove that every convergent sequence is bounded. Justify by an example that the converse is not true.

(b) Prove that the sequence $\langle a_n \rangle$ defined by the recursion formula :

$$a_1 = \sqrt{7}, \; a_{n+1} = \sqrt{7 + a_n}$$

converges to the positive root of $x^2 - x - 7 = 0$.

(c) State Cauchy's convergence criterion for sequences. Check whether the sequence $\langle a_n \rangle$, where

$$a_n = 1 + \frac{1}{5} + \frac{1}{9} + \cdots \cdots\cdots .. + \frac{1}{4n-3}$$

is convergent or not.

(d) Test for convergence the series :

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots\cdots\cdots\cdots\cdots\cdots$$

4. (a) Prove that, if the series $\sum u_n$ converges, then $\lim_{n\to\infty} u_n = 0$. Show by an example that the converse is not true.

(b) Test for convergence the series :

$$\sum_{n=1}^{\infty} \frac{2.4.6 \ldots\ldots\ldots\ldots (2n+2)}{3.5.7 \ldots\ldots\ldots\ldots (2n+3)} x^{n-1} \qquad (x > 0)$$

(c) Let $\langle a_n \rangle$ be a sequence defined by :

$$a_1 = 1,\ a_{n+1} = \frac{3 + 2a_n}{2 + a_n},\ \ n \geq 1.$$

Show that $\langle a_n \rangle$ is convergent and find its limit.

(d) Prove that a sequence of real numbers converges if and only if it is a Cauchy sequence.

5. (a) State Leibnitz test for convergence of an alternating series of real numbers. Apply it to test for convergence the series

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

(b) Show the sequence defined by $\langle a_n \rangle = \langle n^2 \rangle$ is not a Cauchy sequence.

(c) Prove that the sequence $\langle a_n \rangle$ defined by the relation,

$$a_n = 1,\ a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots\cdots\cdots\cdots+\frac{1}{(n-1)!},\quad (n \geq 2),$$

converges.

(d) Prove that every continuous function is integrable.

6. (a) Define Riemann integrability of a bounded function f on a bounded closed interval [a, b]. Show that the function f defined on [a, b] as

$$f(x)=\begin{cases}0 & \text{when } x \text{ is rational}\\ 1 & \text{when } x \text{ is irrational}\end{cases}$$

is not Riemann integrable.

(b) Test for convergence the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos n\alpha}{\sqrt{n^3}},\ \alpha \text{ being real.}$$

(c) State D'Alembert's ratio test for the convergence of a positive term series. Use it to test for convergence the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

(d) Show that $f(x)=\frac{1}{x}$ is not uniformly continuous on $[0,1]$.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5650 **E**

Unique Paper Code : 42344403

Name of the Paper : Computer System Architecture

Name of the Course : **B.Sc. (Prog) / Mathematical Science**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Question No. **1** is compulsory.

3. Attempt any **5** of questions Nos. **2** to **9**.

4. Parts of a question must be answered together.

1. (a) Write the characteristic table of SR flip-flop. (2)

(b) Perform the following operations using signed-2's complement notation for negative numbers in 8-bit representation :

(i) $+42 + (-13)$

(ii) $-42 - (-13)$ (2)

(c) Convert the following numbers to the indicated bases : (2)

(i) $(12121)_3$ to $(----)_{10}$

(ii) $(A675)_{16}$ to $(----)_8$

(d) Differentiate between selective-set and selective-clear. (2)

(e) What is Register? State the use of PC. (2)

(f) Consider the given micro-operation : (2)

$M[AR] \leftarrow AC,\ SC \leftarrow 0$

Write the name of given instruction and state its function.

(g) What is cycle stealing in DMA? (2)

(h) Draw the truth table and logic diagram of Half-Adder. (2)

(i) Specify the output of the following micro-operation : (2)

R3 ← R1 + (R2)' ← 1

(j) Expand the following terms : (2)

(i) CMOS

(ii) ASCII

(iii) TTL

(iv) ECL

(k) Write micro-operations for a following instruction in the basic computer : (2)

LDA (Load to AC)

(l) Construct an 8-to-1 -line multiplexer with two 4-to-1-line multiplexers and one 2 to-1-line multiplexer. Use block diagrams for the three multiplexers. (3)

2. (a) Simplify the following function in Sum-Of-Products (SOP) form using K-map. Also draw the logic diagram.

$F(P, Q, R, S) = \Sigma(0, 2, 5, 7, 8, 10, 11, 12, 14)$

$d(P, Q, R, S) = \Sigma(4, 6)$ (6)

(b) Given the following Boolean function : (4)

$F = A'B + ABC' + ABC$

(i) Simplify the given function F using Boolean algebra.

(ii) Find complement of F using DeMorgan's theorem.

3. (a) A two-word instruction is stored in memory at an address designated by the symbol W. The address field of the instruction (stored at W + 1) is designated by the symbol Y. The operand used during the execution of the instruction is stored at an address symbolized by Z. An index register contains the value X. State how Z is calculated from the other addresses if the addressing mode of the instruction is

(i) direct

(ii) indirect

(iii) indexed (6)

(b) Draw the logic diagram of a 2-to-4-line Decoder with only NOR gates including an enable input. (4)

4. (a) Design a combinational circuit with three inputs a, b, c and three outputs P, Q, R. When the binary input is 0, 1, 2 or 3, the binary output is one greater than the input; otherwise, the binary output is one less than the input. (6)

(b) Obtain the 9's complement of the following 8-digit decimal numbers :

(i) 90009951

(ii) 12349876 (2+2)

5. (a) Explain the functioning of a DMA Controller with the help of a block diagram. (6)

(b) A computer has 32-bit instructions and 12-bit addresses. If there are 250 two-address instructions, how many one-address instructions can be formulated? (4)

6. (a) What is the use of Binary Counter? Draw the 4-bit synchronous binary counter. (2+4)

(b) What is Programmed I/O? Specify any one method that can avoid the drawback of programmed I/O. (4)

7. (a) Design a 4-bit Binary Adder-Subtractor circuit diagram using full-adders. (6)

(b) Consider the following Registers with given values :

$R1 = (00110101)_2$

$R2 = (01100111)_2$

$R3 = (10111001)_2$

$R4 = (11101010)_2$

Determine the 8-bit binary representation of values in each register after the execution of the following sequence of operations. Perform the following operations using R1, R2, R3 and/or R4.

(i) $R1 \leftarrow R1 \oplus R2$

(ii) $R3 \leftarrow R4 - R3$ (4)

8. (a) A computer uses a memory unit with 512K words of 3 2 bits each. A binary instruction code is stored in one word of memory. The instruction has four parts: an indirect bit, an operation code, a register code part to specify one of 64 registers, and an address part. (6)

(i) How many bits are there in the operation code, the register code part, and the address part?

(ii) Draw the instruction word format and indicate the number of bits in each part.

(iii) How many bits are there in the data and address inputs of the memory?

(b) List the micro-operations for Fetch and Decode Phase of Instruction Cycle. (4)

9. (a) Draw the block diagram for the hardware that implements the following :

$$x + yz : AR \leftarrow AR + BR$$

where AR and BR are two n-bit registers and x, y and z are the control variables. Include the logic gates for the control function. (6)

(b) Write a program to evaluate the arithmetic statement :

$$X = (A+B) * (C+D)$$

using two address and three address instructions. (4)

Sl No of QP	: 5713
Unique Paper Code	: 42354401
Name of the Paper	: Real Analysis
Name of the Course	: B.Sc. (Prog) Physical Sciences/Mathematical Sciences
Semester	: IV

Duration : 3 Hours Maximum Marks: 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any two parts from each question.

1. (a) Define a countable set. Show that the set $\mathbb{Q}$ of rational numbers is countable.

(b) Define absolute value of a real number $'x'$.
Find all $x \in \mathbf{R}$ that satisfy the following inequalities:
(i) $4 < |x+2| + |x-1| < 5$
(ii) $|2x-1| \le x+1$

(c) (i) Define supremum of a non-empty bounded subset S of **R**.
(ii) Show that a real number u is the supremum of a non-empty subset S of **R** if and only if it satisfies the following conditions:
(1) $s \le u$ for all $s \in S$.
(2) For each positive real number ε, there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

(6,6)

2. (a) State the Archimedean Property of real numbers. Show that if $x \in \mathbf{R}$, then there exists a unique $n \in \boldsymbol{Z}$ such that $n - 1 \le x < n$.

(b) Define the convergence of a sequence (x_n) of real numbers. Using the definition, evaluate the following limits:

(i) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$
(ii) $\lim_{n\to\infty} \left(\frac{(-1)^n n}{n^2+1}\right)$

(c) Let (x_n) be a sequence of real numbers that converges to x and suppose that $x_n \ge 0$, $\forall n \in \boldsymbol{N}$. Show that the sequence $(\sqrt{x_n})$ converges to $\sqrt{x}$.

(6,6)

3. (a) Prove that every monotonically decreasing and bounded below sequence of real numbers converges.

(b) Show that the sequence (x_n) defined by
$x_1 = 1; x_{n+1} = \frac{1}{4}(2x_n + 3)$, $\forall n \ge 1$ is convergent. Also, find $\lim_{n\to\infty} x_n$.

(c) State Cauchy's Convergence Criterion for sequences of real numbers. Show directly from the definition that the following sequence is a Cauchy sequence:

$$\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right).$$

(6.5, 6.5)

4. (a) State and prove Comparison test for positive term series. Hence, show that the following series converges:

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots$$

(b) Suppose that (x_n) is a sequence of non-negative real numbers. Prove that the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded.

(c) (1) State (without proof) D'Alembert's ratio test for an infinite series.

(2) Test for convergence the series:

(i) $$\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+---.$$

(ii) $$\frac{1}{\log 2}+\frac{1}{(\log 3)^2}+\frac{1}{(\log 4)^3}+\cdots$$

(6.5, 6.5)

5. (a) (i) Define an absolutely convergent series. Is every convergent series absolutely convergent? Justify your answer.

(ii) Test for convergence the series:

(1) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}---$

(2) $\sum_{n=1}^{\infty}(-1)^n.e^{-n}$

(b) Show that if $a > 0$, then the sequence $\left(\frac{nx}{1+n^2x^2}\right)$ converges uniformly on the interval $[a,\infty)$ but not uniformly on the interval $[0,\infty)$.

(c) State Weierstrass M-Test for uniform convergence of series. Hence, show that

$$\sum \frac{1}{x^2+n^2}, \quad \forall x \in \mathbf{R}$$

is uniformly convergent.

(6.5,6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the power series

$$\sum \frac{n+1}{(n+2)(n+3)}x^n.$$

(b) Show that the function $f(x) = x^2$ defined on the interval $[0,b]$, where $b > 0$ is Riemann integrable.

(c) Show that every continuous function defined on $[a,b]$ is Riemann integrable.

(6,6)

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Roll No.

Sr. No. of Question Paper :
Unique Paper Code : 62357603
Name of the Course : B.A. (Prog.)
Name of the Paper : Numerical Methods
Semester : VI
Duration : 3 Hours
Maximum Marks : 75 Marks

Instruction for candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory.
3. Attempt any two parts from each question.
4. Use of non-programmable scientific calculator is allowed.

1 (a) If true value is 0.0003106 and the approximate value is 0.0049065. Then find the absolute and the relative error. Differentiate between round off error and truncation error. [6]

(b) Use the Bisection method to determine the root of the equation

$$x^2 - 3 = 0$$

on the interval [1,2], perform five iterations. [6]

(c) Find a real root of the equation

$$x^3 - 5x + 1 = 0$$

on the interval (0,1) using Secant method to find the root correct up to 3 decimal places. [6]

2 (a) Apply Newton Rapson method to determine the root of the equation

$$x^3 + x^2 - 3x - 3 = 0$$

on the interval [1,2] up to four iterations. [6]

(b) Perform four iterations of the Regula-Falsi method to obtained the real root of the equation

$$x^3 - 2x^2 - 5 = 0$$

on the interval [2,3]. [6]

(c) Derive a formula for finding n^{th} root of a number N, hence find the value of $\sqrt{29}$ using Newton Raphson method. [6]

3(a) Solve the following system of linear equations using Gauss-Elimination method.

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + y + 3z = 14$$

[6.5]

(b) Find the inverse of the coefficient matrix of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

by Gauss-Jordan method with partial pivoting and solve the system. [6.5]

(c) Perform three iterations of Gauss Seidel Method for the following system of equations.

$$4x + y + z = 2$$
$$x + 3y + 2z = -6$$

$$x + 2y + 3z = -4$$

Use initial approximation as $(x, y, z) = (0.5, -0.5, -0.5)$

[6.5]

4(a) Using the following data,

$$f(0) = 1, f(1) = 3, f(3) = 55$$

find the unique polynomial of degree 2 or less which fits the given data. Also, obtain a bound on the error.

[6]

(b) If $f(x) = \frac{1}{x^2}$, find the divided difference of $f[x_1, x_2, x_3, x_4]$

[6]

(c) Obtain the piecewise linear interpolating polynomial for the function $f(x)$ defined by data

X	1.0	2.0	4.0	8.0
f(x)	3	7	21	73

and estimate the values of f(3) and f(7) .

[6]

5(a) Find $f(2)$ *and* $f''(2)$ using quadratic interpolation using the following data:

X	0	1	2	3
f(x)	0	1	4	3

obtain an upper bound on error also.

[6.5]

(b) Approximate the second order derivative of $f(x) = e^x + 2x$ at $x_0 = 0$, taking h = 0.1, 0.01 by using the formula

$$f''(x) \approx \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$

Also approximate the derivative of $f(x) = 1 + x + x^2 + x^3$ at $x_0 = 0$, taking h = 0.1, 0.01 by using the formula

$$f'(x) \approx \frac{f(x_0 + h) - f(x_0 - h)}{h}$$

(c) Integrate the following function $f(x) = \frac{1}{x}$ on the interval [1,2] , using trapezoidal rule with 2,4 and 8 equal subintervals.

[6.5]

6(a) Find the value of

$$\int_2^4 (x^2 + 1)\, dx$$

using Simpson's 1/3rd rule with 2 and 8 equal subintervals. Also, Find the absolute errors.

[6.5]

(b) Consider the initial value problem

$$\frac{dy}{dx} + 2y = 2 - e^{-4x} \quad , y(0) = 1$$

find the approximate value of y(0.5) with step size h=0.1 using Euler Method.

[6.5]

(c) Solve the initial value problem

$$\frac{dy}{dx} + \frac{1}{2}y = 4e^{0.8x} \quad , \; y(0) = 2$$

using the Huen method over the interval [0,0.4] with $h = 0.1$.

[6.5]

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper	**:**	**1271** **F**
Unique Paper Code	:	2352571201
Name of the Paper	:	ELEMENTARY LINEAR ALGEBRA
Name of the Course	:	**B.Sc. (Prog.) DSC-B2**
Semester	:	II

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** question by selecting **two** parts from each question.

3. **All questions** carry equal marks.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that:

$$\|x + y\| \leq \|x\| + \|y\|$$

(b) Define norrn ofa vector. Find a unit vector in the same direction as the vector $\left[\frac{1}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right]$. Is the normalized (resulting) vector longer or shorter than the original? Why?

(c) Use Gaussi.an elimination method to solve föllowing systems of linear equations. Give the complete solution set, and if the solution set is infinite, specify two panicular solutions.

$$\begin{aligned} 3x + 6y - 9z &= 15 \\ 2x + 4y - 6z &= 10 \\ -2x - 3y + 4z &= -6 \end{aligned}$$

2. (a) Determine whether the two matrices are row equivalent?

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$$

(b) Find. the rank of the following matrix.

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}$$

(c) Express the vector X = [2, –5, 3] as a linear combination of the vectors a_1= [1, –3,2], a_2= [2, –4,–1], and a_3= [1, –5,7] if possible.

3. (a) Determine the characteristic polynomial of the following matrix.

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

(b) Show that the set of vectors of the form [a, b, 0, c, a - 2b + c] in $\mathbb{R}^5$ forms a subspace of $\mathbb{R}^5$ under the usual operations.

(c) For S = $\{x^3 + 2x^2, 1 - 4x^2, 12 - 5x^3, x^3 - x^2\}$, use the Simplified Span Method to find a simplified general form for all the vectors in span(S), where S is the given subset of P_3, the set of all polynomials of degree less than or equal to 3 with real coefficients.

4. (a) Use the Independence Test Method to determine whether the given set S is linearly independent or linearly dependent.

s = {1, -1, 0, 2], [0, -2, 1, 0], [2, 0, -1, 1}

(b) Let the subspace W of $\mathbb{R}^5$ be the solution set to the matrix equation AX = 0 where A is

$$\begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -3 & -1 & 1 & 4 \\ 2 & 9 & 4 & -1 & -7 \end{bmatrix}$$

Find the basis and the dimension for W. Show that dim(W) + Rank(A) = 5.

(c) Show that P_n , the set of all polynomials of degree less than or equal to n with real coefficients, is a vector space under the usual operations of addition and scalar multiplication.

5. (a) Consider the mapping $f: R^3 \to R^3$ given by

$$f([a_1, a_2, a_3]) = [a_1, a_2, -a_3]$$

Prove that f is a linear transformation.

(b) Find the matrix for the linear transformation $L: P_3 \to R^3$ given by

$$L(a_3x^3 + a_2x^2 + a_1x + a_0) = [a_0 + a_1, 2a_2, a_3 - a_0]$$

With respect to the bases $B = (x^3, x^2, x, 1)$ for P_3 and $C = (e_1, e_2, e_3)$ for R^3

(c) Consider the linear operator $L: R^n \to R^n$ given by

$$L([a_1, a_2, \cdots, a_n]) = [a_1, a_2, 0 \cdots, 0]$$

Find the kernel of L and range of L.

6. (a) Consider the linear transformation $L: R^3 \to R^3$ given

by $$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the basis for kernel of L.

(b) Consider the linear operator $L: R^2 \to R^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Show that L is one-to-one and onto operator.

(c) Consider the linear transformation $L: P_3 \to P_2$ given by $L(p) = p'$ where $p \in P_3$

Is L an isomorphism?

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1441 **F**

Unique Paper Code : 2352571201

Name of the Paper : Elementary Linear Algebra

Name of the Course : **B.A. (Prog.)**

Semester : II – DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** question by selecting **two** parts from each question.

3. **All** questions carry equal marks.

4. Use of Calculator not allowed.

P.T.O.

1. (a) If x and y are vectors in R^n, then prove that $||x + y|| \leq ||x|| + ||y||$.

Also verify it for the vectors x = [−1, 4, 2, 0, −3] and y = [2, 1, −4, −1, 0] in R^5. (5.5+2)

(b) Prove that for vectors x and y in R^n,

(i) $x . y = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$

(ii) If $(x + y) . (x - y) = 0$, then $||x|| = ||y||$.

(4+3.5)

(c) Solve the systems $AX = B_1$ and $AX = B_2$ simultaneously, where

$$A = \begin{bmatrix} 9 & 2 & 2 \\ 3 & 2 & 4 \\ 27 & 12 & 22 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix}, \text{ and } \quad B_2 = \begin{bmatrix} -12 \\ -3 \\ 8 \end{bmatrix}$$

(7.5)

2. (a) Find the reduced row echelon form of the following matrix :

$$A = \begin{bmatrix} 2 & -5 & -20 \\ 0 & 2 & 7 \\ 1 & -5 & -19 \end{bmatrix}$$ (7.5)

(b) Express the vector x = [2, –1, 4] as a linear combination of vectors v_1 = [3, 6, 2] and v_2 = [2, 10, –4], if possible. (7.5)

(c) Define the rank of a matrix and determine it for the following matrix :

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{bmatrix}$$ (1.5+6)

3. (a) Check if the following matrix is diagonalizable or not :

$$\begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$$ (7.5)

(b) Show that the set of all polynomials P(x) forms a vector space under usual polynomial addition and scalar multiplication. (7.5)

(c) Give an example of a finite dimensional vector space. Check if the following are a vector space or not :

(i) R^2 with the addition $[x, y] \oplus [w, z] = [x + w + 1, y + z - 1]$ and scalar multiplication $a \otimes [x, y] = [ax + a - 1, ay - 2]$.

(ii) set of all real valued functions $f : R \rightarrow R$ such that $f\left(\frac{1}{2}\right) = 1$, under usual function addition and scalar multiplication.

(1.5+3+3)

4. (a) Definesubspace of a vector space. Further show that intersection of two subspaces of a vector space V is a subspace of V. (1.5+6)

(b) Define a linearly independent set. Check if $S = \{(1, -1, 0, 2), (0, -2, 1, 0), (2, 0, -1, 1))$ is linearly independent set in R^4 or not.

(1.5+6)

(c) Define an infinite dimensional and finite dimensional vector space.

Consider the set of all real polynomials denoted by P(x), and the set of all real polynomials of degree at most n denoted by $P_n(x)$. Describe a basis of P(x) and $P_n(x)$ and mention if these are finite dimensional or infinite dimensional.

(2+4+1.5)

5. (a) Show that the mapping $L : M_{nn} \to M_{nn}$, defined as $L(A) = A + A^T$ is a linear operator, where M_{nn} is set of n × n matrices and A^T denotes the transpose of the matrix A. Find the Kernel of L.

(3+4.5)

(b) Let $L: R^2 \to R^3$ be a linear transformation defined as $L\{[a, b]) = [a - b, a, 2a + b]$. Find the matrix of linear transformation A_{BC} of L, with respect to the basis B = {[1,2], [1,0]} and C = {[1,1,0], [0,1,1], [1,0,1]}.

(7.5)

(c) Let L: V → W, be a linear transformation, then define Ker(L), Range(L). Further show that Ker(L) is a subspace of V and Range(L) is a subspace of W. (1.5+1.5+2.5+2)

6. (a) For the linear transformation L: $R^3 \to R^3$ defined as

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Find Ker(L) and Range(L). (4+3.5)

(b) Let L: V → W be a one-to-one linear transformation. Show that if T is a linearly independent subset of V, then L(T) is a linearly independent subset of W. (7.5)

(c) For the linear transformation L: $R^2 \to R^2$, defined as :

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Find L^{-1}, if it exists. (7.5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1473 **F**

Unique Paper Code : 2352201202

Name of the Paper : DSC : Analytic Geometry

Name of the Course : **Bachelor of Arts**

Semester : II

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each section.

4. **All** questions carry equal marks.

1. (a) Sketch the parabola and label the focus, vertex, and directrix of the following

$$y^2 - 10y - 12x + 61 = 0.$$

P.T.O.

(b) State the reflection property of the hyperbola. Sketch the graph of the hyperbola $25y^2 - 9x^2 = 225$, and label the vertices, foci, and asymptotes.

(c) Find an equation for the ellipse satisfying the given conditions: the ends of the major axis are $(0, \pm 6)$; passes through $(-3, 2)$.

2. (a) Find the equation for the parabola that has its vertex at $(5, -3)$, axis parallel to they-axis and which passes through the point $(10, 2)$.

(b) Identify and sketch the curve $xy = 1$.

(c) Find the angle that the vector $\vec{v} = -\sqrt{3}\hat{i} + \hat{j}$ makes with the positive x-axis.

3. (a) Find the vector component of $\vec{v}$ along $\vec{b}$ and the vector component of $\vec{v}$ orthogonal to $\vec{b}$ where $\vec{v} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$.

(b) Find the area of the triangle with vertices $P(1, 5, -2)$, $Q(0,0,0)$ and $R(7,2,0)$.

(c) Use a scalar triple product to determine whether the vectors $\vec{u} = 5\hat{i} - 2\hat{j} + \hat{k}$, $\vec{v} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{w} = \hat{i} - \hat{j}$ lie in the same plane.

4. (a) Find the parametric equation of the line that passes through the point P(–1, 2, 4) and is parallel to the vector $\vec{v} = 3\hat{i} - 4\hat{j} + \hat{k}$.

(b) Find the direction cosines of a line which is perpendicular to the lines whose direction ratios are 1,2,3; –1,3,5.

(c) Show that the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ and the plane $2x - 2y - 2z + 3 = 0$ are parallel and find the distance between them.

5. (a) Find the equation of the sphere described on the join of the points A(2, –3, 4) and B(–5, 6, –7) as diameter.

(b) Prove that the tangent planes to the cone $lyz + mzx + nxy = 0$ are at right angles to the generators of the cone

$$l^2x^2 + m^2y^2 + n^2z^2 - 2mnyz - 2nlzx - 2lmxy = 0.$$

(c) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-2}{2}$.

6. (a) Find the equations of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touching the plane $z = 0$.

(b) Find the equation of the cone whose vertex is (α, β, γ) and base

$$ax^2 + by^2 = 1, \ z = 0.$$

(c) Find the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generators parallel to the line $x = y = z$.

5823

This paper contains 3 printed pages

Unique Paper Code	: 42357602
Name of the Paper	: DSE- Probability and Statistics
Name of the Course	: CBCS-LOCF: B.SC. Physical Sciences/Mathematical Sciences
Semester	: VI
Duration	: 3 Hours
Maximum Marks	: 75

Instructions for Candidates

1. Write your roll number on the top immediately on receipt of this question paper.
2. Attempt all the six questions.
3. Each question has three parts. Attempt any twp parts from each question.
4. Each part in Question 1, 3, 5 carries 6 marks.
5. Each part in Question 2, 4, 6 carries 6.5 marks.
6. Use of scientific calculator is allowed.

1. a) i) If A and B are events in the sample space S, then prove that :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

ii) Let X be a random variable with Distribution Function F_X . Prove that for $a < b$, $P[a < X \leq b] = F_X(b) - F_X(a)$.

b) Show that $f(x) = \begin{cases} e^{-x} & ; 0 < x < \infty \\ 0 & ; elsewhere \end{cases}$ represents a probability density function. Also calculate P(X > 1)

c) i) Show that if X is a random variable and a, b are constants, then

$$E[(aX + b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i}).$$

ii) Let the probability mass function $P(x)$ be positive at $x = -1, 0, 1$ and zero elsewhere. If $P(0) = \frac{1}{4}$, find $E[X^2]$.

2. a) The random variable X has the probability distribution :

$f(x) = \frac{1}{8}\binom{3}{x}$ $for\ x = 0, 1, 2\ and\ 3.$

Find the moment generating function of this random variable and use it to determine μ_1' and μ_2' , where $\mu_r' = E[X^r]$ is the rth moment about the origin.

b) Show that the mean and the variance of the binomial distribution are $\mu = n\theta$ and $\sigma^2 = n\theta(1-\theta)$.

c) Find the probabilities that a random variable having the standard normal distribution will take on a value:

i) less than 1.30.
ii) less than −0.25.
iii) between 0.45 and 1.30.

3. a) Show that the normal distribution has:

i) a relative maximum at $x = \mu$.

ii) inflection points at $x = \mu - \sigma$ and $x = \mu + \sigma$.

b) Let $f(x_1, x_2) = \begin{cases} 4x_1x_2 & ; 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & ; otherwise \end{cases}$ be the pdf of X_1 and X_2.

Find $P(0 < X_1 < \frac{1}{2}, \frac{1}{4} < X_2 < 1)$ and $P(X_1 < X_2)$.

c) Let the random variable X and Y have joint probability mass function as:

(x, y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$P(x, y)$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

i) Find marginal probability mass function of X and Y.

ii) $P(X + Y \leq 2)$.

4. a) Let X_1 and X_2 have the joint pdf $f(x_1, x_2) = \begin{cases} 2 & ; 0 < x_1 < x_2 < 1 \\ 0 & ; otherwise \end{cases}$

i) Find conditional pdf $f_{1|2}(x_1|x_2)$.

ii) Find conditional mean $E(X_1|x_2)$ and the conditional variance $Var(X_1|x_2)$.

b) Let $f(x/y) = \begin{cases} \frac{cx}{y^2} & ; 0 < x < y, 0 < y < 1 \\ 0 & ; elsewhere \end{cases}$

be the conditional density of (X, Y) and

$f_2(y) = \begin{cases} ky & ; 0 < y < 1 \\ 0 & ; elsewhere \end{cases}$ be the marginal pdf of Y. Determine:

i) the constants c and k.

ii) joint pdf of X and Y.

iii) $P\left[\frac{1}{4} < X < \frac{1}{2} \middle| Y = \frac{5}{8}\right]$.

iv) $P\left[\frac{1}{4} < X < \frac{1}{2}\right]$

c) Consider a random experiment that consists of drawing at random one chip from a bowl containing 10 chips of the same shape and size. Each chip has an ordered pair of numbers on it: one with (1, 1), one with (2, 1), two with (3, 1), one with (1, 2), two with (2, 2), and three with (3, 2). Let the random variables X_1 and X_2 be defined as the respective first and second values of the ordered pair. Find the joint probability mass function $p(x_1, x_2)$ of X_1 and X_2, provided with $p(x_1, x_2)$ equal to zero elsewhere.

5. a) If two random variables X and Y have the joint density given by

$$f(x,y) = \begin{cases} e^{-x-y} & ; \quad x > 0, y > 0 \\ 0 & ; \quad elsewhere \end{cases}$$

Show that $M(t_1, t_2) = (1 - t_1)^{-1}(1 - t_2)^{-1}; t_1, t_2 < 1$.

Also show that $[e^{tx+ty}] = (1 - t)^{-2}; t < 1$.

b) Using method of least squares to fit a straight line for the following data:

X	1	2	3	4	5
Y	5	7	9	10	11

c) If X and Y have Joint pdf-

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 ; & otherwise \end{cases}$$

Find the correlation coefficient between X and Y.

6. a) If X and Y have Joint pdf

$$f(x,y) = \begin{cases} 3x & ; 0 < y < x, \; 0 < x < 1 \\ 0 & ; \quad otherwise \end{cases}$$

Are X and Y are independent? If not, find $E[Y|X]$.

b) If $X_i ; i = 1, 2, 3, 4, \ldots, 10$ be independent random variables, each being uniformly distributed over $(0, 1)$. Estimate $P(\sum_1^{10} X_i > 7)$.

c) A die is thrown 3600 times, show that the number of sixes lies between 550 and 650 is at least $\frac{4}{5}$.