Semester-VI

BMATH613: Complex Analysis

Total Marks: 150 (Theory: 75, Internal Assessment: 25 and Practical: 50)
Workload: 4 Lectures, 4 Practicals (per week), **Credits:** 6 (4+2)
Duration: 14 Weeks (56 Hrs. Theory + 56 Hrs. Practical) **Examination:** 3 Hrs.

Course Objectives: This course aims to introduce the basic ideas of analysis for complex functions in complex variables with visualization through relevant practicals. Emphasis has been laid on Cauchy's theorems, series expansions and calculation of residues.

Course Learning Outcomes: The completion of the course will enable the students to:

i) Learn the significance of differentiability of complex functions leading to the understanding of Cauchy–Riemann equations.
ii) Learn some elementary functions and valuate the contour integrals.
iii) Understand the role of Cauchy–Goursat theorem and the Cauchy integral formula.
iv) Expand some simple functions as their Taylor and Laurent series, classify the nature of singularities, find residues and apply Cauchy Residue theorem to evaluate integrals.

Unit 1: Analytic Functions and Cauchy–Riemann Equations
Functions of complex variable, Mappings; Mappings by the exponential function, Limits, Theorems on limits, Limits involving the point at infinity, Continuity, Derivatives, Differentiation formulae, Cauchy–Riemann equations, Sufficient conditions for differentiability; Analytic functions and their examples.

Unit 2: Elementary Functions and Integrals
Exponential function, Logarithmic function, Branches and derivatives of logarithms, Trigonometric function, Derivatives of functions, Definite integrals of functions, Contours, Contour integrals and its examples, Upper bounds for moduli of contour integrals,

Unit 3: Cauchy's Theorems and Fundamental Theorem of Algebra
Antiderivatives, Proof of antiderivative theorem, Cauchy–Goursat theorem, Cauchy integral formula; An extension of Cauchy integral formula, Consequences of Cauchy integral formula, Liouville's theorem and the fundamental theorem of algebra.

Unit 4: Series and Residues
Convergence of sequences and series, Taylor series and its examples; Laurent series and its examples, Absolute and uniform convergence of power series, Uniqueness of series representations of power series, Isolated singular points, Residues, Cauchy's residue theorem, residue at infinity; Types of isolated singular points, Residues at poles and its examples.

Reference:

1. Brown, James Ward, & Churchill, Ruel V. (2014). *Complex Variables and Applications* (9th ed.). McGraw-Hill Education. New York.

Additional Readings:

i. Bak, Joseph & Newman, Donald J. (2010). *Complex Analysis* (3rd ed.). Undergraduate Texts in Mathematics, Springer. New York.
ii. Zills, Dennis G., & Shanahan, Patrick D. (2003). *A First Course in Complex Analysis with Applications*. Jones & Bartlett Publishers, Inc.
iii. Mathews, John H., & Howell, Rusell W. (2012). *Complex Analysis for Mathematics and Engineering* (6th ed.). Jones & Bartlett Learning. Narosa, Delhi. Indian Edition.

Practical / Lab work to be performed in Computer Lab:

Modeling of the following similar problems using Mathematica/Maple/MATLAB/Maxima/ Scilab etc.

1. Make a geometric plot to show that the n^{th} roots of unity are equally spaced points that lie on the unit circle $C_1(0) = \{z : |z| = 1\}$ and form the vertices of a regular polygon with n sides, for $n = 4, 5, 6, 7, 8$.
2. Find all the solutions of the equation $z^3 = 8i$ and represent these geometrically.
3. Write parametric equations and make a parametric plot for an ellipse centered at the origin with horizontal major axis of 4 units and vertical minor axis of 2 units. Show the effect of rotation of this ellipse by an angle of $\frac{\pi}{6}$ radians and shifting of the centre from (0,0) to (2,1), by making a parametric plot.
4. Show that the image of the open disk $D_1(-1 - i) = \{z : |z + 1 + i| < 1\}$ under the linear transformation $w = f(z) = (3 - 4i)z + 6 + 2i$ is the open disk:
$$D_5(-1 + 3i) = \{w: |w + 1 - 3i| < 5\}.$$
5. Show that the image of the right half plane $\text{Re } z = x > 1$ under the linear transformation $w = (-1 + i)z - 2 + 3i$ is the half plane $v > u + 7$, where $u = \text{Re}(w)$, etc. Plot the map.
6. Show that the image of the right half plane $A = \{z : \text{Re } z \geq \frac{1}{2}\}$ under the mapping $w = f(z) = \frac{1}{z}$ is the closed disk $\overline{D_1(1)} = \{w : |w - 1| \leq 1\}$ in the w- plane.
7. Make a plot of the vertical lines $x = a$, for $a = -1, -\frac{1}{2}, \frac{1}{2}, 1$ and the horizontal lines $y = b$, for $b = -1, -\frac{1}{2}, \frac{1}{2}, 1$. Find the plot of this grid under the mapping $w = f(z) = \frac{1}{z}$.
8. Find a parametrization of the polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$, where C_1 is the line from: $-1 + i$ to -1, C_2 is the line from: -1 to $1 + i$ and C_3 is the line from $1 + i$ to $3 - i$. Make a plot of this path.
9. Plot the line segment 'L' joining the point $A = 0$ to $B = 2 + \frac{\pi}{4}i$ and give an exact calculation of $\int_L e^z dz$.
10. Plot the semicircle 'C' with radius 1 centered at $z = 2$ and evaluate the contour integral $\int_C \frac{1}{z-2} dz$.
11. Show that $\int_{C_1} z dz = \int_{C_2} z dz = 4 + 2i$ where C_1 is the line segment from $-1 - i$ to $3 + i$ and C_2 is the portion of the parabola $x = y^2 + 2y$ joining $-1 - i$ to $3 + i$. Make plots of two contours C_1 and C_2 joining $-1 - i$ to $3 + i$.

12. Use ML inequality to show that $\left|\int_C \frac{1}{z^2+1} dz\right| \leq \frac{1}{2\sqrt{5}}$, where C is the straight line segment from 2 to 2 + i. While solving, represent the distance from the point z to the points i and $-i$, respectively, i.e. $|z - i|$ and $|z + i|$ on the complex plane $\mathbb{C}$.
13. Show that $\int_C \frac{dz}{2z^{1/2}}$, where $z^{1/2}$ is the principal branch of the square root function and C is the line segment joining 4 to 8 + 6i. Also plot the path of integration.
14. Find and plot three different Laurent series representations for the function $f(z) = \frac{3}{2+z-z^2}$, involving powers of z.
15. Locate the poles of $f(z) = \frac{1}{5z^4+26z^2+5}$ and specify their order.
16. Locate the zeros and poles of $g(z) = \frac{\pi \cot(\pi z)}{z^2}$ and determine their order. Also justify that Res(g, 0) = $-\pi^2/3$.
17. Evaluate $\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz$, where $C_1^+(0)$ denotes the circle $\{z : |z| = 1\}$ with positive orientation. Similarly evaluate $\int_{C_1^+(0)} \frac{1}{z^4+z^3-2z^2} dz$.

Note: For practicals: Sample materials of files in the form Mathematica/Maple 2011.zip, www.jblearning.com/catalog/9781449604455/.

Teaching Plan (Theory of BMATH613: Complex Analysis):

Week 1: Functions of complex variable, Mappings, Mappings by the exponential function.
[1] Chapter 2 (Sections 12 to 14).

Week 2: Limits, Theorems on limits, Limits involving the point at infinity, Continuity.
[1] Chapter 2 (Sections 15 to 18).

Week 3: Derivatives, Differentiation formulae, Cauchy-Riemann equations, Sufficient conditions for differentiability.
[1] Chapter 2 (Sections 19 to 22).

Week 4: Analytic functions, Examples of analytic functions, Exponential function.
[1] Chapter 2 (Sections 24 and 25) and Chapter 3 (Section 29).

Week 5: Logarithmic function, Branches and Derivatives of Logarithms, Trigonometric functions.
[1] Chapter 3 (Sections 30, 31 and 34).

Week 6: Derivatives of functions, Definite integrals of functions, Contours.
[1] Chapter 4 (Sections 37 to 39).

Week 7: Contour integrals and its examples, upper bounds for moduli of contour integrals.
[1] Chapter 4 (Sections 40, 41 and 43).

Week 8: Antiderivatives, proof of antiderivative theorem.
[1] Chapter 4 (Sections 44 and 45).

Week 9: State Cauchy–Goursat theorem, Cauchy integral formula.
[1] Chapter 4 (Sections 46 and 50).

Week 10: An extension of Cauchy integral formula, Consequences of Cauchy integral formula, Liouville's theorem and the fundamental theorem of algebra.
[1] Chapter 4 (Sections 51 to 53).

Week 11: Convergence of sequences, Convergence of series, Taylor series, proof of Taylor's theorem, Examples.
[1] Chapter 5 (Sections 55 to 59).

Week 12: Laurent series and its examples, Absolute and uniform convergence of power series, uniqueness of series representations of power series.
[1] Chapter 5 (Sections 60, 62, 63 and 66).

Week 13: Isolated singular points, Residues, Cauchy's residue theorem, Residue at infinity.

[1]: Chapter 6 (Sections 68 to 71).

Week 14: Types of isolated singular points, Residues at poles and its examples.
[1] Chapter 6 (Sections 72 to 74).

Facilitating the Achievement of Course Learning Outcomes

Unit No.	Course Learning Outcomes	Teaching and Learning Activity	Assessment Tasks
1.	Learn the significance of differentiability of complex functions leading to the understanding of Cauchy–Riemann equations.	(i) Each topic to be explained with illustrations. (ii) Students to be encouraged to discover the relevant concepts. (iii) Students to be given homework/assignments. (iv) Discuss and solve the theoretical and practical problems in the class. (v) Students be encouraged to apply concepts to real world problems.	• Presentations and class discussions. • Assignments and class tests. • Student presentations. • Mid-term examinations. • Practical and viva-voce examinations. • End-term examinations.
2.	Learn some elementary functions and valuate the contour integrals.		
3.	Understand the role of Cauchy–Goursat theorem and the Cauchy integral formula.		
4.	Expand some simple functions as their Taylor and Laurent series, classify the nature of singularities, find residues and apply Cauchy Residue theorem to evaluate integrals.		

Keywords: Analytic functions, Antiderivatives, Cauchy–Riemann equations, Cauchy–Goursat theorem, Cauchy integral formula, Cauchy's inequality, Cauchy's residue theorem, Closed contour, Contour integrals, Fundamental theorem of algebra, Liouville's theorem, Morera's theorem, Poles, Regions in complex plane, Residue, Singular points, Taylor's and Laurent's series.